Weak transverse isotropy by perturbation theory
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Summary

By requiring a direction independence of wave propagation in a transversely isotropic (TI) system, I arrive at the three, well-known, conditions which govern the passage from a TI system to an isotropic system. This analysis provides the three parameters $\varepsilon$, $\gamma$ and $\eta$, which I use in my application of perturbation theory to TI systems. The first two parameters are the same as those of Thomsen (1986). However, there are differences between the third parameter $\eta$ given here, and its counterpart $\delta$. For weak anisotropy $\delta = 2\eta$.

Next, I apply the non-degenerate perturbation theory to cases when $c_{33} = c_{44}$, provides phase and group velocity expressions for weak and degenerate TI systems. Wave front plots, from these results, are in good agreement with their exact counterparts.

Introduction

Most TI media encountered in geophysical work deviate only slightly from an isotropic state. Thus, to handle the long wave properties of weak TI systems, Thomsen (1986) introduced three anisotropy parameters $\varepsilon$, $\gamma$ and $\delta$, which are small for most crustal rocks. In terms of these three parameters, Thomsen obtained power series approximations for many physical quantities of interest from the exact solutions to Christoffel equations. Among other things, they included first-order approximations for the phase velocities, and the polarization vectors.

Rommel (1994) reinvestigated the polarization of plane waves in weak TI media. He obtained power series expansions of the exact normalized polarization vectors in terms of Thomsen's parameters. Rommel found that the first-order term in the polarization vector reveals the influence of anisotropy.

In this paper, I first examine the conditions for which the propagation of sound waves in TI systems is independent of direction. This analysis yields the three well-known isotropy limits of TI media. By definition, for weak anisotropy, these three relations must deviate from zero only slightly. By strictly using this definition of weak anisotropy, I arrive at these three anisotropy parameters $\varepsilon$, $\gamma$ and $\eta$. The first two parameters are the same as Thomsen's, however, there are differences between the third parameter $\eta$ given here, and its counterpart $\delta$. It can be shown that, for weak anisotropy $\delta = 2\eta$. This correspondence rule allows a direct comparison of results given here and previously published work.

Next, I make a formal investigation of weak elastic anisotropy by the method of perturbation theory. The Christoffel equations establish an eigenvalue problem, with the density times the squared phase velocity as the eigenvalue, and the polarization as the eigenvector. The difference between the Christoffel matrix for TI media ($H$), and the Christoffel matrix for isotropic media ($H_0$), gives the perturbation matrix ($V$). Thus, equipped with the eigenvalues and eigenvectors of $H_0$, I use non-degenerate perturbation theory (Baym, 1969) to compute the modifications produced in eigenvalues and eigenvectors by the addition of the perturbation $V$. First-order results from perturbation theory give phase velocities that agree with Thomsen's results, and polarization vectors that agree with Rommel's expressions.

The second-order perturbation results given in this paper, show the characteristic asymmetry in the group velocity folds, evident from the exact solution, and absent in the first-order results.

The application of degenerate perturbation theory to cases when $c_{33} = c_{44}$, provides phase and group velocity expressions for weak and degenerate TI systems. Wave front plots, from these results, are in good agreement with their exact counterparts. Finally, the regime of applicability of perturbation theory for weak TI systems is discussed, and bounds for elastic parameters are obtained.

Passage To Isotropy and Parameter $\eta$

For a TI system with its vertical axis of symmetry along $x_3$, because of cylindrical symmetry, the coordinate system can always be chosen to simply allow for $k_3 = 0$, where $k$ is the wave vector. Then, the Christoffel equations become:

\begin{align*}
(c_{11}k_1^2 + c_{44}k_2^2)u_1 + (c_{13} + c_{44})k_1u_3 &= \rho\omega^2u_1 \tag{1} \\
(c_{12}k_2u_1 + (c_{13}k_1^2 + c_{44}k_3^2)u_3 &= \rho\omega^2u_1 \tag{2} \\
(c_{44}k_2^2 + c_{44}k_3^2)u_2 &= \rho\omega^2u_2 \tag{3}
\end{align*}

where, $\hat{u}$ represents the local displacement of atoms from their equilibrium positions and $\hat{u}$ is a unit polarization vector.

Since the wave propagation is assumed to be in the ($x,z$) plane, let $p$ and $r$ be the direction cosines for $\hat{k}$, none of which vanish. Thus $k_1 = pk$, and $k_2 = rk$. If a pure longitudinal wave exists, similar relations will have to apply to the normalized polarization vector $\hat{u}$. Thus, $u_1 = p$, and $u_2 = r$. Now, substituting these relations into Eqs. (1) and (2), one obtains:

\begin{align*}
(2c_{44} + c_{13} - c_{44})r^2 + c_{11} &= \rho v^2 \tag{4a} \\
(2c_{44} + c_{13} - c_{44})p^2 + c_{33} &= \rho v^2 \tag{4b}
\end{align*}

These equations hold simultaneously for all values of $p$ and $r$, and therefore for all directions, if $c_{33} = c_{44}$ and $R:

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\[ 2c_{44} + c_{13} - c_{33} = 0. \]
These are two of the three conditions for isotropy. A similar analysis for the propagation of \(SH\) waves provides the third condition for passage from a TI system to an isotropic system, which is \(c_{44} - c_{66} = 0\).

To sum things up, by requiring a direction independence of wave propagation, I have arrived at the three conditions which govern the passage from a TI system to an isotropic system. The conditions are, \(c_{11} - c_{33} = 0\), \(c_{44} - c_{66} = 0\), and \(2c_{44} + c_{13} - c_{33} = 0\). For weak anisotropy, by definition, the deviations of these quantities from zero must be small.

Hence, following Thomsen’s approach, we may define three dimensionless parameters for the study of TI systems in general, and weak TI in particular:

\[ \varepsilon = \frac{c_{11} - c_{33}}{2c_{33}} \]
\[ \gamma = \frac{c_{44} - c_{66}}{2c_{44}} \]
\[ \eta = \frac{2c_{44} + c_{13} - c_{33}}{2c_{33}} \]

The first two parameters are the same as Thomsen’s, however, there are differences between the third parameter \(\eta\) given here, and its counterpart \(\delta\). Thomsen’s parameter can be expressed in terms \(\delta\):

\[ \delta = 2\eta + \frac{2c_{33}}{c_{33} - c_{44}} \eta^2 \]

Using the condition (30) given later, for weak anisotropy, the second-order term in Eq. (8) may be dropped, and we have the correspondence relationship between \(\eta\) and \(\delta\):

\[ \delta = 2\eta \]

### Weak TI by Perturbation Theory

I will now treat the problem of weak anisotropy by applying perturbation theory to the Christoffel equations. First-order perturbation theory gives phase velocities that agree with Thomsen’s (1986) results and polarization vectors that agree with Rommel’s (1994) expressions.

The system of Eqs. (1) and (2) can be formally written as an eigenvalue problem, \(H\beta = \epsilon\beta\). Since the eigenvalue \(E = \rho\beta^2\) has units of energy density, \(H\) can be treated as the Hamiltonian operator in the problem. As such, I set up the weak anisotropy problem in the way of perturbation theory, as discussed in standard quantum mechanics texts (Baym, 1969).

Now, write \(H = H_0 + V\), where the unperturbed Hamiltonian \(H_0\) is the Christoffel matrix for isotropy:

\[ H_0 = \begin{bmatrix} c_{33}\sin^2\theta + c_{44}\cos^2\theta & (c_{13} - c_{33})\sin\theta\cos\theta \\ (c_{33} - c_{44})\sin\theta\cos\theta & c_{33}\cos^2\theta + c_{44}\sin^2\theta \end{bmatrix} \]

and the perturbation matrix \(V\) is:

\[ V = 2c_{33}\begin{bmatrix} \epsilon\sin^2\theta & \eta\sin\theta\cos\theta \\ \eta\sin\theta\cos\theta & 0 \end{bmatrix} \]

\(H_0\) has two eigenvalues, \(E_0 = c_{33}\) and \(E_2 = c_{44}\), respectively belonging to a pure longitudinal, and a pure transverse wave. The corresponding normalized eigenvectors are

\[ \vec{u}_1 = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} \quad \text{and} \quad \vec{u}_2 = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} \]

Now, the effect of the perturbation, to first order, is to shift the eigenvalues by an amount equal to the average of the perturbation matrix \(V\) in the unperturbed states of interest. Using the upper index \(T\) to indicate the transpose of a vector, the first-order corrections to the non-degenerate unperturbed eigenvalues can be computed:

\[ E_0^{(1)} = (\vec{u}_0^T V \vec{u}_0) = 2c_{33}(\epsilon\sin^2\theta + 2\eta\sin^2\theta\cos^2\theta) \]

and

\[ E_2^{(1)} = (\vec{u}_2^T V \vec{u}_2) = 2c_{44}(\epsilon - 2\eta)\sin^2\theta\cos^2\theta \]

Adding these corrections to the unperturbed eigenvalues \((E_0 = c_{33}\) and \(E_2 = c_{44}\)), first-order estimates of the eigenvalues of \(H\) are obtained:

\[ \rho V_{\text{eff}}^{(1)} = c_{33} [1 + 2(\epsilon\sin^2\theta + 2\eta\sin^2\theta\cos^2\theta)] \]

and

\[ \rho V_{\text{eff}}^{(1)} = c_{44} \left(1 + \frac{2c_{33}}{c_{44}}(\epsilon - 2\eta)\sin^2\theta\cos^2\theta\right) \]

These results, through the correspondence relation \(\delta \approx 2\eta\), are in agreement with Thomsen’s expressions for phase velocities.

The first-order correction of each polarization vector is a linear combination of the unperturbed states, with mixing coefficients \((\vec{u}_0^T V \vec{u}_0) / (E_0 - E_2)\). Thus, for the computation of polarization states, we need

\[ (\vec{u}_0^T V \vec{u}_0) = c_{33}(\epsilon\sin^2\theta + \eta\cos^2\theta)\sin(2\theta) \]

Now, via the first-order perturbation theory valid for non-degenerate eigenstates \((c_{33} \neq c_{44})\), I obtain:

\[ \vec{u}_0 + \frac{c_{33}}{c_{33} - c_{44}}(\epsilon\sin^2\theta + \eta\cos^2\theta)(\sin 2\theta)\vec{u}_0 \]

and
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\[ \delta_{qsv} = \delta_{0} + \frac{c_{33}}{c_{44}} (\epsilon \sin^{2}\theta + \eta \cos 2\theta) (\sin 2\theta) \delta_{0}\ . \]  \hspace{1cm} (19)

These results, through the correspondence relation \( \delta \Rightarrow 2 \eta \), agree with Rommel's (1994) expressions.

Equipped with the first-order change in the polarization state (Eq. 17), the second-order corrections to the eigenvalues of \( H \) can be found to be proportional to

\[ E^{(2)} = \left[ \frac{c_{33}(\epsilon \sin^{2}\theta + \eta \cos 2\theta) \sin (2\theta)}{c_{33} - c_{44}} \right]^{2} \ . \] \hspace{1cm} (20)

Thus, referring to (15) and (16), the squares of the phase velocities up to second-order in perturbation \( V \) are given by

\[ \rho \nu^{2}_{p} = c_{33} \left[ 1 + 2 (\epsilon \sin^{2}\theta + 2 \eta \sin^{2}\theta \cos^{2}\theta) \right] + E^{(2)} \] \hspace{1cm} (21)

and

\[ \rho \nu^{2}_{sv} = c_{44} \left[ 1 + \frac{2c_{33}}{c_{44}} (\epsilon - 2\eta) \sin^{2}\theta \cos^{2}\theta \right] - E^{(2)} \ . \] \hspace{1cm} (22)

The second-order eigenvalue correction has the sign of \( (c_{33} - c_{44}) \). The effect of the second-order corrections on the shape of the wave fronts revealed in the group velocities will be illustrated later.

**Degeneracies**

From (18) and (19) it is evident that the closer \( c_{33} \) is to \( c_{44} \), the stronger is the effect of anisotropy. In the limit, when \( c_{33} - c_{44} = 0 \), the situation is strictly degenerate, and needs a special treatment. Here, the unperturbed Hamiltonian \( H_{0} \) reduces to \( c_{33} I \), where \( I \) is the identity matrix, and now the objective is to solve for the eigenvalues and the eigenvectors of \( H = c_{33} I + V \). Since we are dealing with a degenerate two-by-two system, the application of degenerate perturbation theory produces, the exact solutions for \( H \). Using any of the methods for solving two-by-two systems, the exact eigenvalues of the degenerate Hamiltonian, \( H = c_{33} I + V \), are found:

\[ \rho \nu^{2} = c_{33} \left[ 1 + \epsilon \sin^{2}\theta \pm \Gamma \right] \ , \] \hspace{1cm} (23)

where \( \Gamma = |\sin \theta| \sqrt{\epsilon^{2} \sin^{2}\theta + 4 \eta^{2} \cos^{2}\theta} \). Hereafter, the plus and minus signs ( \( \pm \) ) correspond to the quasi \( P \) and quasi \( SV \) waves, respectively. The normalized polarization vectors for the degenerate case are

\[ \delta_{qP} = \left[ \frac{\sqrt{\Gamma + 2\Delta}}{2\Gamma} \right] \quad \text{and} \quad \delta_{qSV} = \left[ \frac{\sqrt{\Gamma - 2\Delta}}{2\Gamma} \right] \ , \] \hspace{1cm} (24)

where \( \Delta = \epsilon \sin^{2}\theta \).

For weak anisotropy, (23) gives the following quasi \( P \) wave and quasi \( SV \) wave velocities:

\[ \nu_{p} = \sqrt{\frac{c_{33}}{\rho} \left[ 1 + 0.5 (\epsilon \sin^{2}\theta \pm \Gamma) \right]} \ . \] \hspace{1cm} (25)

For weak and degenerate TI systems, just as the non-degenerate cases discussed by Thomsen, group velocities and phase velocities satisfy the following simple relations, \( V_{p} (\phi) = \nu_{p} (\phi) \). It can also be shown that for weak and degenerate TI systems, the group velocity angle \( \phi \) and the phase velocity angle \( \theta \) satisfy

\[ \phi_{\pm} = \theta + \cos \theta \left[ \epsilon \sin^{2} \pm \text{sgn}(\sin \theta) \Psi (\theta) \right] \ , \] \hspace{1cm} (26)

where

\[ \Psi (\theta) = \frac{\epsilon^{2} \sin^{2}\theta + 2 \eta^{2} \cos 2\theta}{\sqrt{\epsilon^{2} \sin^{2}\theta + 4 \eta^{2} \cos^{2}\theta}} \ . \] \hspace{1cm} (27)

**Regime of Applicability**

Perturbation theory is applicable when the mixing coefficients are much less than one. Thus, referring to Eq. (17), we must have

\[ \left| \epsilon \sin^{2}\theta + \eta \cos 2\theta \right| < \frac{c_{33} - c_{44}}{c_{33}} \ . \] \hspace{1cm} (28)

Since the condition in Eq. (28) depends on the wave propagation angle \( \theta \), it lacks the necessary generality for defining the regime of applicability of weak anisotropy. A less precise, but more manageable condition is obtained by simply requiring \( V \) to be much smaller than \( H_{0} \). This means that the matrix elements of \( V \) are much smaller than those of \( H_{0} \). Thus, referring to Eqs. (10) and (11), we have

\[ \epsilon \ll \frac{1}{2} \left[ 1 + \frac{c_{44}}{c_{33}} \cot^{2}\theta \right] \quad \text{and} \quad \eta \ll \frac{c_{33} - c_{44}}{2c_{33}} \ . \] \hspace{1cm} (29)

For seismic waves, the longitudinal wave speed is, approximately, twice the speed of the transverse wave. Thus, \( c_{33} = 4c_{44} \), and from (30) we may conclude \( \eta \ll 3/8 \). Also, if weak anisotropy analysis is to be valid for all directions of wave propagation, then (29) gives \( \epsilon \ll 1/2 \).

**Discussion of Results**

Figures 1, 2, and 3, show slowness curves and the group velocity curves for \( P \) waves and \( SV \) waves. I used, \( \epsilon = 0.2 \), \( \eta = 0.2 \), and \( c_{33}/c_{44} = 0.25 \). The effect of anisotropy is more pronounced on the structures of folds revealed in the group velocities for \( SV \) modes (Helbig, et al., 1986 and Carcione, et al., 1992). The second-order perturbation results (Fig. 3) show the characteristic asymmetry in the folds of the group velocity as evident from the exact solution
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(Fig. 1) and absent in the first-order results (Fig. 2).

Figures 4 and 5 correspond to degenerate cases when \( c_{33} = c_{44} \). They show that, phase and group velocity (wave front plots) from expressions for weak and degenerate TI systems (Fig 5), are in good agreement with their exact counterparts (Fig 4). Here I used \( \epsilon = 0.2 \) and \( \eta = 0.3 \).

All the curves have been normalized to \( c_{11}/\rho \).

Conclusions

A formal investigation of the problem of weak elastic anisotropy by the method perturbation theory was given. First-order results from non-degenerate perturbation theory give phase velocities that agree with Thomsen's results and polarization vectors that agree with Rommel’s expressions. An advantage of the perturbation approach is that first-order approximations were obtained by working directly with the Christoffel matrix, rather than with its exact solutions. Thus, the present work provides an independent check for the previously published weak anisotropy results, and resolves their differences.

The application of degenerate perturbation theory to cases when \( c_{33} = c_{44} \) provides phase and group velocity expressions for weak and degenerate TI systems. Wave front plots, from these results, are in good agreement with their exact counterparts.

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References