DMO by The Huygens-Fresnel Diffraction Integral
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SUMMARY

In this paper a general expression for the DMO operator is derived, by using the Huygens-Fresnel (H-F) diffraction integral as a representation for the exploding reflector model. The expression obtains both the near field (Fresnel) and the far field (Fraunhofer) diffraction processes associated with the DMO problem. By neglecting the local wavefront curvatures in the general solution, an expression for the far field H-F DMO operator is then obtained. As a further check, the same result is also derived by the use of superposition of beams, and conservation of energy arguments.

Subsequently, the far field H-F DMO operator is formally compared with Hale's F-K DMO operator. It is shown that, for large offset values, the H-F DMO operator reduces to a form which differs from Hale's F-K DMO operator by an overall phase shift of $\pi/2$. An equivalent interpretation based on a high frequency approximation, is also given.

The investigation of the analytic solution of the H-F DMO operator, reveals several key properties. It is shown that this operator, has no real solutions inside the DMO ellipse. Physically, this is consistent with the fact that no waves can arrive earlier than the time of the shortest path. Hence, the H-F DMO operator is strictly causal. Another advantage of the H-F DMO operator is that it is equipped with a source amplitude spectrum. That this can be of practical significance in true amplitude and wavelet processing applications is briefly discussed.

DMO BY THE HUYGENS-FRESNEL INTEGRAL

The basic objective of any DMO operator is to map reflection data from a finite offset, to the corresponding zero offset section. The locus of all possible reflections, for a given source and receiver offset having the same arrival times, is of course an ellipsoid of revolution. The exploding reflector model offers a point of view for constructing the zero offset section from this ellipsoidal reflector. In other words, in this approximation, the zero offset section can be pictured as arising exclusively from a set of fictitious non-interacting point sources distributed uniformly over the ellipsoid. (The source strengths are chosen to be proportional to the value of the reflection coefficient, and are set off simultaneously.)

In view of the exploding reflector model, and in accordance with the Huygens-Fresnel principle, each element on the ellipsoidal reflector (Fig. 1) is envisioned as a continuous emitter of spherical secondary waves. Hence, for monochromatic sources of frequency $\omega$, the total disturbance at an observation point $P$ on the surface, a distance $y$ from the origin $O$, is given by the superposition of these waves (Born p. 380):

$$ U(P) = -i \frac{A_0 e^{-iat}}{2\lambda} \int \frac{e^{ikR}}{R} (1 + \cos \chi) dS. \tag{1} $$

where, $R$ is the distance from the element $dS$ on the reflector and the observation point $P$, and $A_0$ is the amplitude. The wave number, the wavelength, and the frequency, satisfy the usual relationships $k = 2\pi / \lambda$, and $\omega = Vk$. The element $dS$ explores the domain of integration $\Sigma$, which is the surface area of the half ellipsoid of revolution, shown in Fig. 1a. The above is an expression of the Huygens-Fresnel Diffraction formula. The expression includes the actual inclination factor:

$$ K(\chi) = -i \frac{1}{2\lambda} (1 + \cos \chi), \tag{2} $$

where, $\chi$ is the angle made with the normal to the ellipse $P'H$ and the line of observation $PP'$. The inclination has its maximum value in the upward direction. $K(0) = 1$, and also properly dispenses with the back-propagating waves, since $K(\pi) = 0$. Referring to Fig. 1, consider points $H$ along the $y$ axis, where normal-lines emanating from the ellipse, say from $P'$, intersect the $y$ axis.

Now, the two lengths $P'HI$ and $OH$ are the same. Hence, for monochromatic sources of frequency $\omega$, the Huygens-Fresnel principle, each element on the ellipsoidal reflector simultaneously.

To cast Eq. (1) in a useful form, it needs to be expressed in terms of the basic parameters of the ellipse, $a$, $b$, and $\theta$. Furthermore, to make the discussion of the far field approximations given in the next section simpler, the observation distance $R(\theta, y)$ is most naturally expressed in terms of its components along the normals $R_\alpha(\theta)$, and the tangents $R_\gamma(\theta)$ to the ellipse. Thus, referring to Fig. 1b, we have:

$$ R_\gamma(\theta, y) = h(\theta) + (y - y) \sin \theta. \tag{5} $$

and

$$ R_\gamma(\theta, y) = (y - y) \cos \theta, \tag{6} $$

and hence, $R(\theta, y) = \sqrt{R_\alpha^2 + R_\gamma^2}$. Similarly, for the cosine term of the inclination factor in Eq. (2), we have:

$$ \cos(\chi) = \frac{R_\gamma(\theta, y)}{R_\alpha(\theta, y)}. \tag{7} $$

Finally, the element of surface area on the ellipsoid of revolution which contributes to the observation point $P'$ is the shaded strip shown on Fig. 1a, and has the cross-section shown in Fig. 1c. The radius of the revolution is $h(\theta) \cos \theta$, and the surface element itself is:

$$ dS(\theta) = \pi R^2(\theta) \cos \theta d\theta. \tag{8} $$

Substituting these results into Eq. (1), the complete disturbance arriving at $y$ due to a uniform distribution of point sources on the ellipsoid is obtained:

$$ U = -i \frac{\pi}{4V} \int d\omega A(\omega) e^{-ia\omega} \int \frac{e^{ikR(\theta)}}{R(\theta, y)} [1 + \frac{R_\gamma(\theta, y)}{R(\theta, y)}] A(\theta) \cos \theta d\theta. \tag{9} $$

where, $A(\omega)$ is the source spectrum. This is the general Huygens-Fresnel representation of the DMO operator, and to the extent that the exploding
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THE FAR-FIELD (FRAUNHOFER) APPROXIMATION

The DMO operator given in Eq. (9) encompasses both the near field and the far field solutions. When the parameters of the problem (distances compared to wavelengths) are such that the curvature of the wavefronts are not negligible, then Fresnel diffraction prevails. On the other hand, when the wavefronts emerging from the ellipse, locally approach being planar (differing by a small fraction of a wavelength) Fraunhofer diffraction obtains.

By definition, in the exploding reflector model the sources are all in phase. Hence, referring to Fig. 1b, the source distribution on the infinitesimal surface element dS, may be treated as a point source placed at P'.

Now, if

\[ R(\theta, y) = R_N(\theta, y) \leq \frac{\lambda}{2} \]  

(10)

then the two points P, and H' will be essentially in phase. Then, the observation point P is said to be in the far field of the source at P', and Fraunhofer diffraction prevails. In particular, if \( P'P \approx PH' \), then the far field condition for the DMO operation is:

\[ \frac{(y_P - y)^2 \cos^2 \theta}{2(h(\theta) + (y_P - y) \sin \theta)} \leq \frac{\lambda}{2} \]  

(11)

Under these conditions, all \( R(\theta, y) \) in Eq. (9) can be replaced by \( R_N(\theta, y) \). This substitution is made for the phase term in Eq. (9), however, for the amplitude terms the far field is sufficiently approximated by replacing \( R(\theta, y) \) by \( h(\theta, y) \). Upon the substitution of equations (3), and (4) into (5), the normal distance \( R_N(\theta, y) \), is simplified to:

\[ R_N(\theta, y) = \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta - y \sin \theta} \]  

(12)

Thus, the far field approximation of the DMO operator becomes:

\[ U_{HF}(y, t) = -\frac{i}{4V} \int_0^\infty d\omega A(\omega) e^{-i\omega t} \int_{-\omega^2}^{\omega^2} e^{iKR_N(\theta, Y)h(\theta) \cos \theta} d\theta, \]  

(13)

where, \( \omega = \frac{V}{k} \). This is the basic representation of the far field Huygens-Fresnel (H-F) DMO operator. Thus, given the source and receiver offset (2F) and the reflection arrival time (T), along with the medium velocity (V), then the a and b parameters of the ellipse are established. Hence, the zero offset section can be computed from a finite offset by Eq. (13).

To allow for a direct comparison of Hale's F-K DMO operator with the far field H-F DMO operator, Eq. (13) is first expressed in terms of positive frequencies. Then, it is transformed by the use of the horizontal component of the wave number \( k = k_x = k \sin \theta \). As a result of this substitution, the far field H-F DMO operator becomes:

\[ U_{HF}(y, t) = VT \int_0^\infty d\omega A(\omega) \int_{-\omega^2}^{\omega^2} \sin[\omega t - \omega T \Lambda(\omega, \kappa) + \kappa y] d\kappa \]  

(14)

where,

\[ \Lambda(\omega, \kappa) = \sqrt{1 + \left( \frac{Fk_x}{\omega} \right)^2} \]  

(15)

In obtaining (14), a basic property of the ellipse \( a^2 - b^2 = F^2 \), where \( F \) is half the distance between the foci of the ellipse, was also used.

Another useful form of the far field DMO operator is obtained by introducing the following change of variables in the \( \theta \) integral of Eq. (13):

\[ \Phi = \tan^{-1} \left( \frac{\lambda}{2F} \right) \]  

(16)

As a result of this transformation, the far field DMO operator becomes:

\[ U_{HF}(y, t) = \frac{b^2}{2\pi VF} \int_{-\Phi}^{\Phi} dx \int_{-\Phi}^{\Phi} d\omega A(\omega) \omega \sin \left[ \omega \Gamma(x) \right], \]  

(17)

where,

\[ \Gamma(x) = [t - \left( \frac{b}{V} \cos x \right) + \left( \frac{by}{VF} \right) \tan x], \]  

(18)

and

\[ \Phi = \tan^{-1} \left( \frac{\lambda}{2Fb} \right). \]  

(19)
CONNECTIONS WITH HALE’S DMO OPERATOR

The far field H-F DMO operator expressed as in Eq. (17), is comparable with Hale’s F-K DMO operator Eq. (A4). To make the comparison more explicit, let the limits of the z integral, in Eq. (17) go to \( \pi / 2 \). Referring to Eq. (19), this is equivalent to letting \( \Phi \rightarrow \pi / 2 \), which also implies letting \( (F / b) \rightarrow \infty \). Hence the large offset approximation of the DMO operator (17), is obtained:

\[
U_{HF}^{yy}(y,t) = \frac{\lambda}{2\pi VF} \int_{-\pi/2}^{\pi/2} \int_{-\rho}^{\rho} d\rho A(\rho) \cos \theta \sin [\omega T(x)] \, d\theta.
\]  

Aside from the leading constants and an overall phase shift of \( \pi / 2 \), the large offset approximation of the H-F DMO operator is seen to agree with Hale’s F-K DMO operator. For a strict comparison, in Eq. (20) the amplitude spectrum \( A(\omega) \) must be set equal to unity.

In light of this last approximation, referring to Eq. (15), it is noted that when \( F / b = FV / T \) is large, then, except for very large frequencies \( \omega \), the denominator of Eq. (14), \( \Lambda(\omega, \kappa) \), will be a large quantity. Thus, for large offset values, the limits of the \( \kappa \) integral in Eq. (14) can be extended to infinity. That is, let \( (\omega / \nu) \rightarrow \infty \). This permits a change in the integration order, and gives:

\[
U_{HF}^{yy}(y,t) = \frac{\lambda}{\nu} \int_{-\pi/2}^{\pi/2} d\theta \sin [\omega T \Lambda(\omega, \kappa) - \omega t - k y] A(\omega) \Lambda(\omega, \kappa). \tag{21}
\]

Once again, aside from the leading constants, and the \( \pi / 2 \) phase shift, this expression agrees with Hale’s F-K DMO operator given by Eq. (A2).

For a strict agreement with the F-K DMO operator, the amplitude spectrum \( A(\omega) \) in Eq. (21) must be set equal to unity. The presence of the source amplitude spectrum offers a practical advantage in applications involving true amplitude and wavelet processing. It also prevents the emergence of indefinite terms in the analytic solution of the F-K DMO operator (Ohanian, 1993).

DMO BY SUPERPOSITION OF BEAMS

The far field H-F DMO operator can be derived, directly, by the use of two physical arguments: (a) close to the exploding reflector, wave vectors \( \vec{k} \) are perpendicular to the ellipse, and hence, the emerging waves can be treated as beams of plane waves. (b) The energy carried away by each beam is conserved during the wave propagation.

Referring to the exploding reflector model shown in Fig. 1(a), consider the waves generated by the point sources on the shaded circular strip. Let the width of the strip be \( D \). Because of the finite width of this strip, the point sources on it will not generate true plane waves. Instead, such a strip will emit "beams" of waves, as depicted in Fig. 2(a). A beam is like a segment of a plane wave, in the sense that, all waves in the beam travel in the same approximate direction. In the course of its propagation a beam spreads out, and its amplitude decreases. This is because, the radiated energy spreads out over a larger area. This point of view will now be used to estimate the observed wave-field.

Referring to Fig. 1(a), the average energy emitted by the sources on the shaded circular strip of width \( D \) is:

\[
\text{Source Beam Energy} = A_s^2 D \pi h(\theta) \cos \theta \tag{22}
\]

where, \( A_s \) is the average source amplitude.

Consider the two dimensional view of things shown in Fig. 2(b). Given \( D \) as the spatial width of the beam at the reflector, then there will be an angular distribution of propagation direction with a "full width at half maximum intensity" of about \( \lambda / D \) (Landau and Lifshitz p. 144). By the time the beam has traveled a distance \( h(\theta) \) from the reflector, it will spread out in its lateral dimension to an amount, given by \( \lambda h(\theta) / D \).

Now, referring to fig. 2b, it is noted that on the y axis itself where we are recording the waves, the beam will have an apparent spread of:

\[
W_y = 2 \pi h(\theta) / k D \cos \theta. \tag{23}
\]

Although in planes normal to the y axis the narrow shaded strip (Fig. 1a) follows a perfect circle, still, the rays converging into the observation point \( P \) will not focus perfectly. That is, in light of the uncertainty principle, the width of the image in the x direction will not be zero. Rather, there will be a definite smear along the x axis. This smear is in the order of a wavelength:

\[
W_x \approx 1 / k. \tag{24}
\]

Thus, the energy due to each circular strip on the ellipsoid will cross a rectangle of area \( (W_x W_y) \), surrounding the observation point \( P \). Thus:

\[
\text{Observed Beam Energy} = A_s^2 D \pi \hbar \tag{25}
\]

Setting the source and the observed beam energies equal, and using \( D = h(\theta) \cos \theta \), then the average amplitude at the observation point is obtained:

\[
A = A_s \hbar \tag{26}
\]

Now, the total field due to all beams of a single frequency \( \omega \), emerging from the exploding reflector is a superposition of them all:

\[
U_{HF}^{yy}(y,t) = A_s e^{i \omega \nu} \int_{-\pi/2}^{\pi/2} \int_{-\rho}^{\rho} d\rho \sin [\omega T \Lambda(\omega, \kappa) - \omega t - k y] A(\omega) \Lambda(\omega, \kappa). \tag{27}
\]

Now, in view of Eq. (12), the exponent of the integrand in Eq. (27) is \( \vec{k} \cdot \vec{r} = k (r_x \cos \theta, r_y \sin \theta) \). After multiplying the above result by \( i \), and

\[
\text{Fig. 2a Superposition of Beams}
\]

\[
\text{Fig. 2b Normally Emerging Wavefronts}
\]
integrating over all frequencies, the expression for the far field H-F DMO operator Eq. (13) is obtained. Note that the presence $i$ in Eq. (13) insures its realness.

It is interesting to note that, if Eq.(27) is integrated over positive frequencies only, then its real part gives Hale’s F-K DMO operator.

**ANALYTIC PROPERTIES OF THE H-F DMO OPERATOR**

When the source amplitude spectrum is white, the H-F DMO operator given in (13) permits an analytic closed solution. The solution is obtained by directly transforming the Eq. (13), via the variable changes defined in Eq. (16):

$$U^\text{HF}(y,t) = \frac{ib^2}{VF} \int_{0}^{\infty} \frac{dx}{\cos x} \int_{-\infty}^{\infty} d\omega \omega e^{-i\omega \Gamma(x)},$$  \hspace{1cm} (28)

where, $\Gamma(x)$ was defined in Eq. (18). Introducing the multiplicative parameter $\alpha$ as a coefficient to the $b/V \cos x$ in the exponent $\Gamma(x)$, the above expression can be written as a derivative of a delta function:

$$U^\text{HF}(y,t) = -\lim_{\alpha \rightarrow 1} \frac{b}{F} \frac{\partial}{\partial \alpha} \int_{0}^{\infty} \int_{-\infty}^{\infty} \delta(x) \delta(\alpha \cos x - \frac{by}{V \tan x}).$$  \hspace{1cm} (29)

Now, the following property of the delta function of a function $g(x)$ is invoked:

$$\delta(g(x)) = \sum_{i} \int_{[g'(x_i)]} \delta(x-x_i).$$  \hspace{1cm} (30)

where, $g(x_i) = 0$, and $g'(x_i) \neq 0$. Thus, the delta function integral in (29) can be simply carried out. The result is then differentiated with respect to $\alpha$, and the limit of $\alpha \rightarrow 1$ is taken. An analytic closed form for the H-F DMO operator is then obtained. The details of this will be given elsewhere. The point of interest here is that function $g(x)$ in the delta function of Eq. (29), has two real roots. They exist only when:

$$\left(1 + \frac{t^2}{F^2}\right) \geq 1.$$  \hspace{1cm} (31)

Hence, the far field H-F DMO operator has no solutions inside the DMO ellipse. Solutions exist on the ellipse, and the outside of the ellipse. Physically, this is consistent with the fact that no waves can arrive from the reflector ellipse to surface observation points, earlier than the time of the shortest path. These paths are along the normals to the ellipse and define the DMO ellipse. In short, the H-F DMO operator developed here, is strictly causal. The analytic solution to Hale’s F-K DMO operator (Ohanian, 1993) has non zero solutions outside, on, and inside the DMO ellipse.

**CONCLUSIONS**

The formulation of the DMO problem by the Huygens-Fresnel diffraction integral provides the following conclusions:

- The H-F DMO operator is equipped with a source amplitude spectrum which can be useful in true amplitude and wavelet processing applications.
- The exploding reflector model considered here, is an ellipsoid of revolution. Hence, the DMO analysis given in this paper, correctly accounts for the three dimensional amplitude attenuation, due to the spherical expansion of wave fronts.

**REFERENCES**

Ohanian V., 1993, Analytic properties of DMO operators, SEG Washington

**APPENDIX: A REVIEW OF HALE’S DMO OPERATOR**

Consider a seismic experiment with a shot and receiver offset of 2\(F\), conducted in a constant velocity medium over an arbitrarily oriented reflector. After NMO correction, let the energy impulse occur at the time 2\(T\). According to Hale’s 1984 DMO formulation, the zero offset section associated with this experiment is given by:

$$U^H(t,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega d\kappa \frac{\delta_\omega(\omega, \kappa - y)}{\Lambda(\omega, \kappa)},$$  \hspace{1cm} (A1)

where, $\Lambda(\omega, \kappa)$ is given by Eq. (15). Now, utilizing the fact that $\Lambda(\omega, \kappa)$ is an even function of $\omega$, and $\kappa$, (A1) can be expressed in the following form:

$$U^H(t,y) = 2 \int_{0}^{\infty} \int_{-\infty}^{\infty} d\omega \cos[\omega T \Lambda(\omega, \kappa) - \omega t - \kappa y].$$  \hspace{1cm} (A2)

Introducing the following change of variables in the above expression,

$$\kappa = \left(\frac{T\omega}{F}\right) \tan x,$$  \hspace{1cm} (A3)

the following form of Hale’s DMO operator, relevant to present discussion, is obtained

$$U^H(t,y) = \frac{2T}{F} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\omega \omega \cos[\omega \Gamma(x)],$$  \hspace{1cm} (A4)

where, $\Gamma(x)$ was given in Eq. (18).